

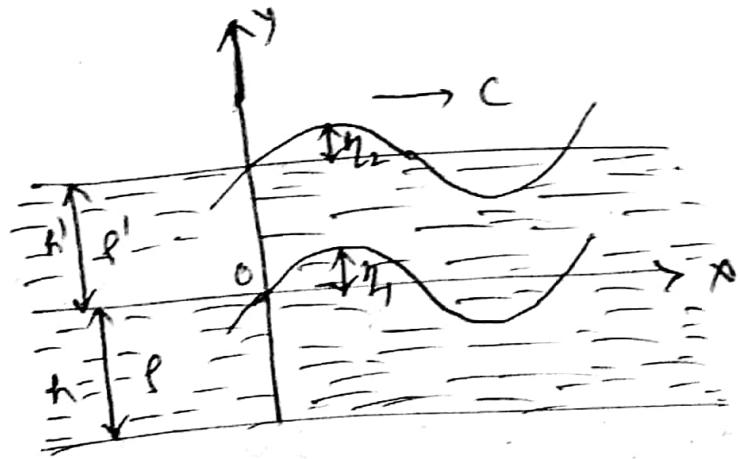
Ex. An infinite liquid of density  $\sigma$  lies above an infinite liquid of density  $\rho$ , the two liquids being separated by a horizontal plane of interface. Show that the velocity  $v$  of propagation of waves of length  $\lambda$  along the interface is given by

$$v^2 = \frac{g\lambda}{2\pi} \left( \frac{\rho - \sigma}{\rho + \sigma} \right) \quad \text{provided } \rho > \sigma.$$

Ans. [case III above].

### § 1.10. Waves at an interface with upper surface free.

Let a liquid of density  $\rho'$  and depth  $h'$  lies over another liquid of density  $\rho$  and depth  $h$  and let the liquids



be at rest except for wave motion. If the system be slightly disturbed then the waves at the common surface and free surface can be propagated. Let us suppose a common velocity  $c$  of wave profile at the free surface of the upper liquid and at the common surface. we

choose the axes as shown in figure.

To make the motion steady, we super-impose a velocity  $-c$  on the whole system so that the wave profile is reduced to rest and liquid begins to move with velocity  $c$  in negative direction of  $x$ -axis.

The complex potential of lower and upper liquids are

$$W = cz + \frac{ac}{\sinh m\eta} \cos m(\bar{\eta} + it) \quad \text{--- (1)}$$

$$\text{and } W' = cz - \frac{ac}{\sinh m\eta'} \cos m(\bar{\eta} - it') + \frac{bc}{\sinh m\eta'} \cos m\bar{\eta} \quad \text{--- (2)}$$

The first expression is written from the consideration of steady motion and for the second we had to replace  $t$  by  $-t'$  in (1) besides the addition of the term  $\frac{bc \cos m\bar{\eta}}{\sinh m\eta'}$

which represents the complex potential of a simple sine wave  $\eta_2 = b \sin mx$  at time  $t=0$ . In fact (2) is obtained from the superposition of (1).

Let  $q^L$  be the fluid velocity in the <sup>lower</sup> liquid.

$$\text{Then } q^L = \frac{dw}{d\bar{\eta}} \cdot \frac{d\bar{w}}{d\bar{\eta}}$$

$$= \left[ c - \frac{acm}{\sinh m\eta} \sin m(\bar{\eta} + it) \right] \cdot \left[ c - \frac{acm}{\sinh m\eta} \sin m(\bar{\eta} - it) \right]$$

$$= c^2 \left[ 1 - \frac{am}{\sinh m\eta} \{ \sin m(\bar{\eta} + it) + \sin m(\bar{\eta} - it) \} \right] - \left[ \begin{matrix} \text{neglecting} \\ a^2 \end{matrix} \right]$$

$$\text{or } q^L = c^2 \left[ 1 - \frac{2am}{\sinh m\eta} \sin mx \cosh m(y + t) \right] \quad \text{--- (3)}$$

For the speed in the upper liquid, we have

$$q'^U = \frac{dw'}{d\bar{\eta}} \cdot \frac{d\bar{w}'}{d\bar{\eta}}$$

$$= \left[ c + \frac{acm}{\sinh m\eta'} \sin m(\bar{\eta} - it') - \frac{bcm}{\sinh m\eta'} \sin m\bar{\eta} \right]$$

$$\cdot \left[ c + \frac{acm}{\sinh m\eta'} \sin m(\bar{\eta} + it') - \frac{bcm}{\sinh m\eta'} \sin m\bar{\eta} \right]$$

since  $a, b$  are small quantities and hence neglecting  
 $a^2, ab, b^2$ , we obtain

$$q'^2 = c^2 \left[ 1 + \frac{am}{\sinh mR_1} \left\{ \sin m(\bar{x}-it) + \sin m(\bar{x}+it) \right\} - \frac{bm}{\sinh mR_1} \left\{ \sin m\bar{x} + \sin m\bar{x} \right\} \right]$$

$$\text{or } q'^2 = c^2 \left[ 1 + \frac{2am}{\sinh mR_1} \sin mx \cosh m(y-t) - \frac{2bm}{\sinh mR_1} \sin mx \cdot \sinh my \right] \quad (4)$$

In order to get the values of velocities  $q'_0$  and  $q_0$  due to upper and lower liquids at the common surface we put  $\eta_1 = a \sin mx, y=0$  in (3) and (4) and get

$$q_0^2 = c^2 \left[ 1 - 2m\eta_1 \coth mR_1 \right]$$

$$q'_0^2 = c^2 \left[ 1 + 2m\eta_1 \coth mR_1 - \frac{2mb\eta_1}{a \sinh mR_1} \right]$$

Pressure equation for steady motion, gives

$$\frac{p}{\rho} + \frac{1}{2} q_0^2 + g\eta_1 = \text{const.}$$

$$\text{and } \frac{p'}{\rho'} + \frac{1}{2} q'_0^2 + g\eta_1 = \text{const.} \text{ at the common surface.}$$

But  $p=p'$  at the common surface. Hence on subtraction,

$$\frac{1}{2} (p' q'_0^2 - p q_0^2) + g\eta_1 (p' - p) = \text{const.}$$

$$\text{or } g\eta_1 (p' - p) + p' c^2 \left( \frac{1}{2} + m\eta_1 \coth mR_1 - \frac{mb}{a} \cdot \frac{\eta_1}{\sinh mR_1} \right) - p c^2 \left( \frac{1}{2} - m\eta_1 \coth mR_1 \right) = \text{const.}$$

This equation holds for every value of  $\eta_1$  and so coefficient of  $\eta_1$  must vanish.

Thus we have,

$$g(p' - p) + mc^2 \left\{ p' \coth mR_1 + p \coth mR_1 - p' \left( \frac{b}{a} \right) \operatorname{cosech} mR_1 \right\} = 0 \quad (5)$$

since  $p'$  is constant at the free surface

$$y = h' + b \sin mx = h' + \eta_2,$$

Then pressure equation

$$\frac{p'}{\rho'} + \frac{1}{2} g'^2 + gy = \text{const. becomes } \frac{1}{2} g'^2 p' + g p' y = \text{const.}$$

$$\text{or } gp'(h' + b \sin mx) + \rho' c^2 \left\{ \frac{1}{2} + \frac{am}{\sin mh'} - b m \sin mx \coth mh' \right\} = \text{const.}$$

(by putting  $y = h' \sin mx$ )

As before, coefficient of  $\sin mx$  must vanish so that

$$bg + c^2 \cdot \left( \frac{am}{\sin mh'} - b m \coth mh' \right) = 0$$

$$\text{or } \frac{b}{a} = \frac{mc^2}{\sin mh' (c^2 m \coth mh' - g)}$$

$$= \frac{c'm}{c'm \coth mh' - g \sin mh'} \quad \text{--- (6)}$$

putting the value of  $\frac{b}{a}$  from this equation into the equation (5),

$$g(p-p') + mc^2 (p' \coth mh' + p \coth mh') - \frac{mc^4 p' (\coth mh' - 1)}{c'm \coth mh' - g} = 0.$$

Simplifying, we get -

$$c^4 m^2 (p \coth mh' \cdot \coth mh' + p') - c^2 mg p (\coth mh' + \coth mh') + g^2 (p-p') = 0 \quad \text{--- (7)}$$

This eqn gives the possible velocity of propagation for a given wavelength whenever  $p \neq p'$ .  
Corollary: If the lower liquid is deep, then  $h \rightarrow \infty$  so

that  $\coth mh = 1$ . Then (7) becomes,

$$c^4 m^2 (p \coth mh' + p') - c^2 mg p (1 + \coth mh') + g^2 (p-p') = 0$$

$$\text{or } (mc^2 - g) \{ mc^2 (p \coth mh' + p') - g (p-p') \} = 0$$

$$\Rightarrow c^2 = \frac{g}{m} \text{ and } c^2 = \frac{g(p-p')}{m(p \coth mh' + p')}$$

Ex If there be two liquids in a straight canal of uniform sections of densities  $\sigma_1, \sigma_2$  and depths  $l_1, l_2$ , show that the velocity  $c$  of propagation of long waves is given by the eqn

$$\left(\frac{c^2}{l_1 g} - 1\right) \left(\frac{c^2}{l_2 g} - 1\right) = \frac{\sigma_1}{\sigma_2}$$

where  $\sigma_2 > \sigma_1$  and it is assumed that the liquids do not mix.

Sol: Since the two liquids are contained in a straight canal and so the surface of the upper liquid must be free surface.

For this, we have

$$c^4 m^2 (\ell \coth m h \cdot \coth m h' + p)$$

$$- c^2 m g p (\coth m h + \coth m h') + g^2 (p - p') = 0$$

Here,  $p = \sigma_2$ ,  $p' = \sigma_1$ ,  $h = l_2$ ,  $h' = l_1$ , as  $p > p'$ , we have

$$\sigma_2 > \sigma_1.$$

$$\text{Then } c^4 m^2 (\sigma_2 \coth m l_2 \cdot \coth m l_1 + \sigma_1)$$

$$- c^2 m g \sigma_2 (\coth m l_2 + \coth m l_1) + g^2 (\sigma_2 - \sigma_1) = 0$$

Since the waves formed at the common surface are long waves so that

$$m l_1 \rightarrow 0, m l_2 \rightarrow 0 \quad (\text{For } \frac{h}{\lambda} = \frac{m h}{2\pi} \rightarrow 0)$$

Consequently  $\tanh m l_1 = m l_1$ ,  $\tanh m l_2 = m l_2$ .

$$\therefore c^4 m^2 \left( \sigma_2 \cdot \frac{1}{m l_1} \cdot \frac{1}{m l_2} + \sigma_1 \right) - c^2 m g \sigma_2 \left( \frac{1}{m l_2} + \frac{1}{m l_1} \right) + g^2 (\sigma_2 - \sigma_1) = 0$$

$$\text{or } \frac{c^4}{l_1 l_2} + \frac{\sigma_1}{\sigma_2} \cdot c^4 m^2 - c^2 g \left( \frac{1}{l_1} + \frac{1}{l_2} \right) + g^2 \left( 1 - \frac{\sigma_1}{\sigma_2} \right) = 0$$

$$\text{But } m^2 = \left( \frac{2\pi}{\lambda} \right)^2$$

In case of long waves,  $\lambda$  is very large so that  $m^2$

is negligible.

Then the last eqn becomes

$$\frac{c^4}{l_1 l_2 g^2} - \frac{c^2}{g} \left( \frac{1}{l_1} + \frac{1}{l_2} \right) + 1 = \frac{\sigma_1^2}{\sigma_2^2}$$

or  $\left( \frac{c^2}{l_1 g} - 1 \right) \left( \frac{c^2}{l_2 g} - 1 \right) = \frac{\sigma_1^2}{\sigma_2^2}$  proved

### 8.1.11 Group velocity:

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When waves are started by a local disturbance such as dropping of stone into a canal or the motion of a boat through water, the successive waves have different lengths and are propagated with different velocities. Since the velocity of propagation of a simple harmonic wave varies with the wave length so the waves of slightly different wave lengths will be sorted out into groups. The velocity with which an isolated group of waves of considerably the same length, advances over relatively deep water is called the group velocity.

We consider the case of two systems of simple harmonic waves of the same amplitude and of nearly but not quite the same wave length such that

$$\eta_1 = a \sin(mx - nt) \quad \text{--- (i)}$$

$$\eta_2 = a \sin \{(m+\delta m)x - (n+\delta n)t\} \quad \text{--- (ii)}$$

where  $\delta m$  and  $\delta n$  are small quantities.

The resultant disturbance is given by

$$\begin{aligned}\eta &= \eta_1 + \eta_2 \\ &= a \sin(mx - nt) + a \sin \left[ (m+\delta m)x - (n+\delta n)t \right] \\ &= 2a \cos \frac{x \cdot \delta m - t \cdot \delta n}{2} \cdot \sin \left\{ \left( m + \frac{\delta m}{2} \right)x - \left( n + \frac{\delta n}{2} \right)t \right\}\end{aligned}$$

$$= A \sin(m'x - n't) \quad \text{--- (iii)}$$

where  $A = 2a \cos \frac{x\delta m - t\delta n}{2}$ ,  $m' = m + \frac{\delta m}{2}$ ,  $n' = n + \frac{\delta n}{2}$

Hence The resultant wave is of the same form (iv)  
as one of the original waves but with a different  
amplitude. \* From (iv), it follows that the amplitude  
of the resultant disturbance (iii) varies as a wave  
velocity  $C_g = \frac{\delta n}{\delta m}$ , known as group velocity.

In differential notation

$$C_g = \frac{dn}{dm} \quad \boxed{\begin{array}{l} \text{(iii) shows that the resulting disturbance} \\ \text{is a progressive sine wave whose} \\ \text{amplitude } A \text{ is not constant but} \\ \text{is itself varying as a wave} \\ \text{velocity } C_g = \frac{\delta n}{\delta m}. \text{ This velocity} \\ \text{is known as the group} \\ \text{velocity.} \end{array}}$$

But the wave velocity is  $C = \frac{1}{m}$

$$\therefore n = cm$$

$$\text{Hence } C_g = \frac{d}{dm}(cm)$$

$$= c \cdot 1 + m \frac{dc}{dm} \quad \text{--- (v)}$$

Again for the waves on the surface of water of depth  $h$ , we have the relation

$$c^2 = \frac{n^2}{m^2}$$

$$\text{or } c^2 = \frac{g}{m} \tanh mh$$

Taking log on both sides, neglect

$$2 \log c = \log g - \log m + \log \tanh h$$

Differentiating w.r.t.  $m$ , we have

$$\frac{2}{c} \cdot \frac{dc}{dm} = -\frac{1}{m} + h \cdot \frac{\operatorname{sech}^2 mh}{\tanh mh}$$

$$= -\frac{1}{m} + h \cdot \frac{1}{\operatorname{cosech}^2 mh} \cdot \frac{\operatorname{cosech} mh}{\tanh mh}$$

$$= -\frac{1}{m} + \frac{2h}{\sinh^2 mh}$$

$$\text{or } m \frac{de}{dm} = -\frac{c}{2} + \frac{cmh}{\sinh 2mh}$$

Thus (v) becomes

$$C_g = c + m \cdot \frac{de}{dm}$$

$$= c - \frac{c}{2} + \frac{cmh}{\sinh 2mh}$$

$$= \frac{c}{2} + \frac{cmh}{\sinh 2mh}$$

$$= \frac{c}{2} \left[ 1 + \frac{2mh}{\sinh 2mh} \right]$$

i.  $\frac{C_g}{c} = \frac{1}{2} \left[ 1 + \frac{2mh}{\sinh 2mh} \right] \quad (\text{vi})$

Case I: For deep water,  $h \rightarrow \infty$  so that

$$\frac{2mh}{\sinh 2mh} \rightarrow 0$$

Hence (vi) becomes

$$\frac{C_g}{c} = \frac{1}{2}$$

$$\begin{aligned} & \frac{2mh}{\sinh 2mh} \xrightarrow{\text{Form } \frac{0}{0}} \\ &= \lim_{h \rightarrow 0} \frac{2m}{\tanh 2mh} \\ &= \lim_{h \rightarrow 0} \frac{1}{e^{2mh} + e^{-2mh}} \\ &= \frac{1}{e^0 + e^0} = \frac{1}{2+0} = 0 \end{aligned}$$

$$\Rightarrow C_g = \frac{c}{2}$$

i. The group velocity for deep water is half the wave velocity.

case II: For shallow water,  $h \rightarrow 0$  so that

$$\frac{2mh}{\sinh 2mh} \rightarrow 1$$

$\therefore$  (vi) becomes

$$\frac{C_g}{c} = \frac{1}{2} [1+1]$$

$$\Rightarrow C_g = c$$

i. The group velocity for shallow water is equal to the wave velocity. //